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Hyperbolicity of the complement of generic quartic plane curves

Based on joint work with

- X. Chen & E. Riedl '22 (CRY)
- K. Ascher & A. Turchet '24 (ATY)

Curves

X smooth proj. curve $/ \mathbb{Q}$

$g(X)$	$\# X(\mathbb{Q})$	$\deg \overset{2g-2}{\overset{\shortparallel}{K_X}}$	Entire curves $\mathbb{C} \to X(\mathbb{C})$
0	Potentially dense	< 0	$\exists\, \mathbb{C} \to \mathbb{P}^1 \to X(\mathbb{C})$
1	Potentially dense	$= 0$	$\exists\, \mathbb{C} \to X(\mathbb{C})$
≥ 2	Finite Faltings '83	> 0	$\nexists\, \mathbb{C} \xrightarrow{\text{const}} \Delta \to X(\mathbb{C})$

U smooth affine curve $/\mathbb{Q}$

$\cap$

X smooth proj. curve, $D = X \setminus U$ snc divisor

$g(X)$	$\#D$	$U(\mathbb{Z})$	entire curves $\mathbb{C} \to U(\mathbb{C})$	$\deg K_X + D$
0	≤ 2	Potentially dense		≤ 0
	≥ 3	Finite Siegel '29	$\nexists$ by Little Picard	> 0
≥ 1				

(X, D) is hyperbolic if $2g(X) - 2 + \#D > 0$

Higher dimensions $/\mathbb{C}$

(X, D) smooth log pair

X smooth proj. var

D snc divisor

$U := X \setminus D$

1. $\mathcal{U}$ is <u>Brody hyperbolic</u> if $\nexists$ entire curves $\mathbb{C} \to \mathcal{U}$.

2. (X, D) is <u>algebraically hyperbolic</u> if $\exists \varepsilon > 0$ $\exists H$ ample divisor on X s.t. $\forall f: Y \to X$ with $f(Y) \not\subset D$, (Y: smooth proj. curves)

X. Chen
Demailly

$$2g(Y) - 2 + |f^{-1}(D)| > \varepsilon \deg_H Y.$$

Examples

1. $\dim X = 1$: Brody hyperbolic = alg hyperbolic

2. X = abelian variety Bloch, Kawamata, Ochiai
   $D \subset X$ ample divisor not containing translates of abelian subvarieties

   $X \setminus D$ Brody hyperbolic

3. $X = \mathbb{P}^n$, D very general, smooth degree d hypersurface

Kobayashi's conjecture 70's

- $d \geq 2n+1 \Rightarrow (X, D)$ alg hyp X. Chen 00s, Pacienza and Rousseau
- $d \geq$ deg-4 poly in n $\Rightarrow X \setminus D$ Brody hyp Berczi and Kirwan '22
- $d \leq 2n \Rightarrow \exists$ bicontact lines to $D \Rightarrow$ not hyp.

4. OR $\begin{cases} \exists \text{ rational curves in } X \text{ intersecting } D \text{ in } \leq 2 \text{ points} \\ \exists \text{ elliptic curves } \text{———} \text{ " } \text{———} \ 1 \text{ point} \end{cases}$

$\Rightarrow$ AND $\begin{cases} (X, D) \text{ not alg hyp.} \\ X \setminus D \text{ not Brody hyp} \end{cases}$

(Alg)

Green-Griffiths-Lang conjecture:

(X, D) log general type — $K_X + D$ big

$\Rightarrow (X, D)$ alg hyp modulo proper subvariety $S \subsetneq X$.

"pseudo hyperbolic"

Plane curves $X = \mathbb{P}^2$, $D =$

1. 5 lines in gen. pos. $\Rightarrow$ hyperbolic.

   Dufresnoy, Green: Same holds for $\geq 2n+1$ hyperplanes
   '44 in gen. pos. in $\mathbb{P}^n$.

2. k components of degrees $d_1, \dots, d_k$ in gen. pos. and snc

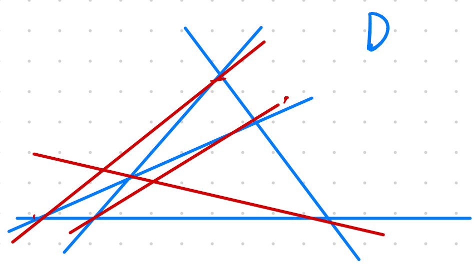
$$\begin{cases} k \geq 5 \\ k = 4 \text{ AND } \sum d_i \geq 5 \\ k = 3 \text{ AND } d_i \geq 2 \end{cases} \Rightarrow \text{Brody hyp}$$

Green '77
Babets '84
Dethloff, Schumacher, Wong '95

3. 4 lines in gen. pos.

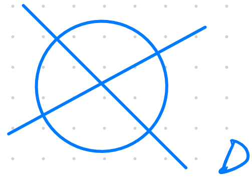
   $\mathbb{P}^2 \setminus D$ Brody hyp modulo

   3 diagonal lines.



Bloch, Cartan '20s: The complement of $n+2$ hyperplanes in gen. pos. in $\mathbb{P}^n$ is Brody hyp modulo "diagonal hyperplanes."

4. Conic + 2 lines in gen. pos.



Corvaja, Zannier, Turchet: $(\mathbb{P}^2, D)$ alg hyp modulo $S \subsetneq \mathbb{P}^2$

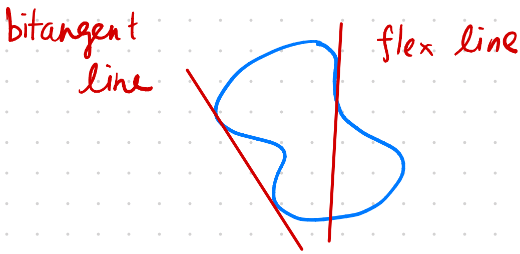
Caporaso, Turchet '24: There are only finitely many curves $C \subset \mathbb{P}^2$ with $|C \cap D| \leq 2$, and they are rational.

5. Very general quartic plane curves with 1 component

CRY '22: $(\mathbb{P}^2, D)$ alg hyp modulo bicontact lines

$$2g(C) - 2 + |C \cap D| \geq \frac{1}{2} \deg C.$$

$\exists$ conics in $\mathbb{P}^2$ intersecting D in 3 pts.



6. Very general snc quartic plane curves w/ 2 components

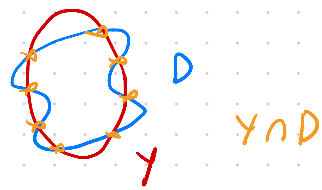
ATY '24: $(\mathbb{P}^2, D)$ alg hyp modulo finitely many curves

$$2g(C) - 2 + |C \cap D| \geq \frac{1}{2} \deg C.$$

Log vector fields

$(\mathbb{P}^2, D)$ smooth pair

$f : Y \to \mathbb{P}^2$



<u>Definition</u> (log tangent sheaves)

$$0 \to T_{\mathbb{P}^2}(-\log D) \to T_{\mathbb{P}^2} \to \mathcal{O}_D(D) \to 0$$

Local sections of $T_{\mathbb{P}^2}(-\log D)$ are local vector fields on $\mathbb{P}^2$ that are tangent to D.

$(Y, Y \cap D)$

$$0 \to T_Y(-\log D) \to T_Y \to \mathcal{O}_{Y\cap D}(Y \cap D) \to 0$$

<u>S.e.s.</u>

$$0 \to T_Y(-\log D) \to f^* T_{\mathbb{P}^2}(-\log D) \to \underbrace{N_{f/\mathbb{P}^2}(\log D)}_{\text{log normal sheaf}} \to 0$$

Versal families

Consider B_0 = param. space for hypersurfaces. $H^0(\mathcal{O}_{\mathbb{P}^2}(4))$

Spread $Y, D \subset \mathbb{P}^2$ into versal families: $H^0(\mathcal{O}_{\mathbb{P}^2}(d_1)) \oplus H^0(\mathcal{O}_{\mathbb{P}^2}(d_2))$

$$\begin{tikzcd}
 & \mathcal{Y} \arrow[dr] & & \\
\mathcal{X} = \mathbb{P}^2 \times B \supset & & B \arrow[r, "\text{étale}"] & B_0 \\
 & \mathcal{D} \arrow[ur] & &
\end{tikzcd}$$

$(\mathcal{X}, \mathcal{D})$ smooth pair

$$0 \to T_{\mathcal{Y}}(-\log \mathcal{D}) \to T_{\mathcal{X}}(-\log \mathcal{D})|_{\mathcal{Y}} \to N_{\mathcal{Y}/\mathcal{X}}(\log \mathcal{D}) \to 0$$

<u>Theorem</u> $T_{\mathcal{X}}(-\log \mathcal{D}) \otimes p^*\mathcal{O}_{\mathbb{P}^2}(1)$ glob. gen.

$\Rightarrow T_{\mathbb{P}^2}(-\log D)(1)$ glob. gen.

$\Rightarrow N_{f/\mathbb{P}^2}(\log D)(1)$ glob. gen.

$$\Rightarrow \deg N_{f/\mathbb{P}^2}(\log D)(1) \geq 0$$

$$\deg N_{f/\mathbb{P}^2}(\log D)(1) = 2g-2+|f^{-1}(D)| - \deg f^*(K_{\mathbb{P}^2}+D) + \deg f^*\mathcal{O}_{\mathbb{P}^2}(1)$$

$$\Rightarrow 2g-2+|f^{-1}(D)| \geq (d-4)\deg f^*\mathcal{O}_{\mathbb{P}^2}(1).$$

Vertical log tangent sheaves

$$0 \to T_X^{vert}(-\log D) \to T_X(-\log D) \to p^* T_{\mathbb{P}^n} \to 0$$

Kernel bundles $0 \to M_d \to H^0(\mathcal{O}(d)) \otimes \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(d) \to 0$

At point $p \in \mathbb{P}^n$, $M_d|_p$ = deg-d polys vanishing at p.

Lemma For $P \in H^0(\mathcal{O}_{\mathbb{P}^n}(d-1))$, $M_1 \xrightarrow{\cdot P} M_d$

There is a surjection $M_1^{\oplus s} \twoheadrightarrow M_d$

Theorem ① One degree d component:

CRY

$$p^* M_d \underset{\mathcal{O}_X}{\subset} T_X^{vert}(-\log D)$$

② Two components of degrees d_1 and d_2:

ATY

$$p^* M_{d_1} \oplus p^* M_{d_2} \underset{\mathcal{O}_X}{\subset} M_{d_1, d_2} \underset{\mathcal{O}_X}{\subset} T_X^{vert}(-\log D)$$

Theorem

① One degree d component: CRY

There is a surjection

$$\begin{tikzcd} T_X^{vert}(-\log D)|_y \arrow[r, two heads] & N_{g/X}(\log D) . \\ M_d|_y \arrow[u, phantom, "\cup" rotate=90] \arrow[ur, two heads] & \end{tikzcd}$$

$$\Rightarrow \quad M_1^{\oplus s}|_y \twoheadrightarrow M_d|_y \twoheadrightarrow N_{f/\mathbb{P}^2}(\log D)$$

rank $N_{f/\mathbb{P}^2}(\log D) = 1$

$$\Rightarrow \quad M_1|_y \xrightarrow{m_p} N_{f/\mathbb{P}^2}(\log D) \qquad p \in H^0(\mathcal{O}(d-1))$$

$$\deg N_{f/\mathbb{P}^2}(\log D) \geq \deg(\text{image of } M_1|_y)$$

$$\geq -\deg f^*\mathcal{O}_{\mathbb{P}^2}(1) .$$

② Two components of degrees d_1 and d_2: ATY

There is a surjection

$$M_1^{\oplus s}|_y \twoheadrightarrow M_{d_1} \oplus M_{d_2}|_y \twoheadrightarrow N_{y/X}(\log D)$$